Transonic Nozzle Flows of Gases with a Rate Process

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Theme

IN the aerodynamic design of a convergent-divergent supersonic nozzle, the transonic flow within the throat region must be known to provide the flow conditions on a supersonic start line from which the supersonic design can be initiated. Therefore, many efforts have already been devoted to solving the transonic flow near the geometric throat in the nozzle. Almost all the attempts, however, have been concerned with the potential flow of a classical ideal gas.

From a practical point of view, it is quite significant to investigate the transonic flow with rate processes and furthermore with nonuniform flow properties across the flowfield. In this paper, the uniform transonic flow is first considered, and then the flow with nonuniform properties across the flowfield is investigated analytically. Only axially symmetric flows are considered, and analytic solutions are based on a series expansion in terms of the wall geometry in the transonic region and are obtained for both equilibrium and frozen flows. The result given here is in essense an extension of the solution first obtained by Hall.[1]

Contents

It is convenient for our purpose to use the general description of the streamline and the orthogonal trajectory to the streamlines

$$\alpha(x,y) = \text{const} \qquad \text{along} \quad \frac{dy}{dx} = -\cot\theta \tag{1}$$

$$\beta(x,y) = \text{const} \qquad \text{along} \quad \frac{dy}{dx} = \tan\theta \tag{2}$$

where x is the distance along the nozzle axis, y the distance from the axis, and θ the flow angle relative to the x axis. If the nozzle profile near the throat $f(x)$ is assumed to be a parabolic, hyperbolic, or circular arc, it can be described as follows,

$$f(x) = r + \frac{x^2}{2R} + O\left(\frac{x^4}{R^3}\right) \tag{3}$$

where R is the radius of curvature of the nozzle contour and r the half height at the throat. It is assumed that the quantity $\epsilon = r/R$ is small compared with unity and can be taken as a perturbation parameter.

Since we consider only frozen and equilibrium flows, the flow variables at the critical (sonic) point on each streamline are uniquely determined from the stagnation conditions. Now the dimensionless quantities are introduced by

$$\bar x = x/L \qquad \bar y = y/r \qquad \bar\rho = \rho/\rho_* \qquad \bar V = V/a_{i*}$$

$$\bar p = p/p_* \qquad \bar q = q/q_* \qquad \bar a_i = a_i/a_{i*} \tag{4}$$

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where L is a reference length and taken to be $L = \epsilon^{1/2} r$, and ρ, V, p, q, and a are, respectively, the density, the velocity, the pressure, the progress variable, and the speed of sound. The subscript i is f or e, which denotes the frozen or equilibrium flow, and the asterisk denotes the critical (sonic) conditions. The quantities ρ_*, a_*, p_*, and q_* may not be constants but functions of β.

Transforming the independent variables from (x,y) to (α,β) and using the dimensionless quantities introduced above, we can describe the system of basic equations for an inviscid, nonconducting flow as follows

$$\frac{1}{\bar\rho\bar V}\frac{\partial}{\partial\alpha}(\bar\rho\bar V) - \epsilon^{-1/2}\frac{(\partial\bar y/\partial\alpha)}{(\partial\bar x/\partial\beta)}\frac{\partial\theta}{\partial\beta} + \frac{1}{\bar y}\frac{\partial\bar y}{\partial\alpha} = 0 \tag{5}$$

$$\bar\rho\bar V\frac{\partial\bar V}{\partial\alpha} + \frac{1}{\gamma_i}\frac{\partial\bar p}{\partial\alpha} = 0 \tag{6}$$

$$\bar\rho\bar V^2\frac{\partial\theta}{\partial\alpha} - \epsilon^{-1/2}\frac{1}{\gamma_i}\frac{(\partial\bar y/\partial\alpha)}{(\partial\bar x/\partial\beta)}\frac{\partial\bar p}{\partial\beta} = \epsilon^{-1/2}\frac{1}{\gamma_i}\frac{(\partial\bar y/\partial\alpha)}{(\partial\bar x/\partial\beta)}\bar p\frac{\partial}{\partial\beta}(\ln p_*) \tag{7}$$

$$\frac{\partial\bar p}{\partial\alpha} - \gamma_i\bar a_i^2\frac{\partial\bar\rho}{\partial\alpha} = 0 \tag{8}$$

$$\frac{\partial\bar y}{\partial\alpha} = \epsilon^{1/2}\tan\theta\frac{\partial\bar x}{\partial\alpha} \tag{9}$$

$$\frac{\partial\bar y}{\partial\beta} = -\epsilon^{1/2}\cot\theta\frac{\partial\bar x}{\partial\beta} \tag{10}$$

where $\gamma_i = a_{i*}^2/(p_*/\rho_*)$, which is not always equal to the ratio of the specific heats. Equations (9) and (10) are given from Eqs. (1) and (2), respectively. It must be noticed that if the independent variables α and β are newly introduced by Eqs. (9) and (10), they will be dimensionless to begin with. An analytic solution is sought to such a flow for the nozzle

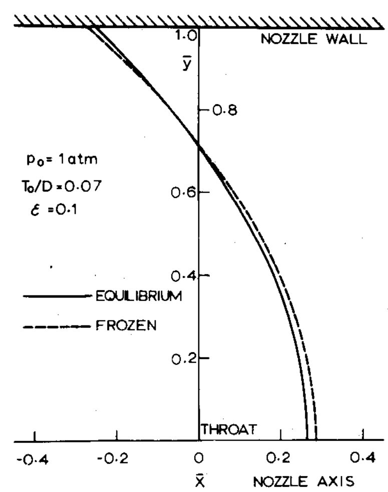

Fig. 1 Sonic lines for equilibrium and frozen flows.

geometry given by Eq. (3) utilizing a perturbation method similar to that first used by Hall.[1]

In the case of uniform flows, the right-hand side of Eq. (7) vanishes and the solution can be easily constructed as follows:

$$\bar{V}=1+\epsilon\left\{m_i\alpha+\frac{1}{2}\beta^2-\frac{1}{4}\right\}+\epsilon^2\left\{-\frac{1}{6}n_im_i^4\alpha^2+\frac{1}{2}m_i(\beta^2-\frac{1}{4})\alpha+\frac{1}{8}(\frac{7}{2}+\frac{1}{3}n_im_i^2)\beta^4-(\frac{3}{4}+\frac{1}{12}n_im_i^2)\beta^2+(\frac{3}{16}+\frac{5}{288}n_im_i^2)\right\}+O(\epsilon^3) \quad (11)$$

$$\theta=\epsilon^{3/2}\left\{\alpha\beta+\frac{1}{4m_i}\beta(\beta^2-1)\right\}+\epsilon^{5/2}\left\{(\frac{3}{4}+\frac{1}{6}n_im_i^2)\alpha\beta(\beta^2-1)+\frac{1}{12m_i}(\frac{15}{4}+\frac{2}{3}n_im_i^2)\beta(\beta^4-1)-\frac{1}{16m_i}(\frac{21}{2}+\frac{5}{3}n_im_i^2)\beta(\beta^2-1)\right\}+O(\epsilon^{7/2}) \quad (12)$$

$$\bar{x}=\alpha+\epsilon\left\{-\frac{1}{2}\alpha(\beta^2-1)-\frac{1}{16m_i}(\beta^2-1)^2\right\}+\epsilon^2\left\{-(\frac{3}{16}+\frac{1}{24}n_im_i^2)\alpha(\beta^2-1)^2-\frac{1}{72m_i}(\frac{15}{4}+\frac{2}{3}n_im_i^2)(\beta^2-1)^3+\frac{1}{192m_i}(\frac{3}{2}-\frac{1}{3}n_im_i^2)(\beta^2-1)^2\right\}+O(\epsilon^3) \quad (13)$$

$$\bar{y}=\beta+\epsilon^2\left\{\frac{1}{2}\alpha^2\beta+\frac{1}{4m_i}\alpha\beta(\beta^2-1)\right\}+O(\epsilon^3) \quad (14)$$

$$\bar{\rho}=2-\bar{V}+\epsilon^2(1-\frac{1}{m_i^2})\left\{m_i\alpha+\frac{1}{2}\beta^2-\frac{1}{4}\right\}^2+O(\epsilon^3) \quad (15)$$

$$\bar{p}=1+\gamma_i-\gamma_i\bar{V}+O(\epsilon^3) \quad (16)$$

where

$$m_i=(\gamma_iA_{i10}+A_{i01}+1)^{-1/2}$$

$$n_i=\frac{1}{m_i^2}(\frac{3}{m_i^2}-4)+2(\frac{1}{m_i^2}-1)A_{i01}-(\gamma_i^2A_{i20}+2\gamma_iA_{i11}+A_{i02}) \quad (17)$$

and

$$A_{i\nu\lambda}=\left[\frac{\partial^{\nu+\lambda}}{\partial\bar{p}^\nu\partial\bar{\rho}^\lambda}\bar{a}_i(\bar{p},\bar{\rho})\right]_{\bar{p}=\bar{\rho}=1}\quad \left.\begin{matrix}\nu\\ \lambda\end{matrix}\right\}=0,1,2 \quad (18)$$

It is important to notice that the forms of the solutions to the frozen and equilibrium flows are the same. From the solution, the sonic line location in the physical plane can easily be determined as follows

$$\bar{x}=-\frac{1}{2m_i}(\bar{y}^2-\frac{1}{2})+\frac{\epsilon}{m_i}\left\{\frac{1}{8}(\frac{5}{2}+\frac{1}{3}n_im_i^2)\bar{y}^2-\frac{1}{16}(\frac{3}{2}+\frac{1}{9}n_im_i^2)\right\}+O(\epsilon^3) \quad (19)$$

which represents a parabolic arc. Also the mass flow rate M and the thrust F are obtained as follows

$$\frac{M_i}{\pi r^2}=\rho_*a_{i*}\left\{1-\frac{\epsilon^2}{48m_i^2}+O(\epsilon^3)\right\} \quad (20)$$

$$\frac{F_i}{\pi r^2}=\frac{\gamma_i+1}{\gamma_i}\rho_*a_{i*}^2\left\{1-\frac{\epsilon^2\gamma_i}{48(\gamma_i+1)m_i^2}+O(\epsilon^3)\right\} \quad (21)$$

It can be verified in general that for all the internally relaxing and chemically reacting gases $\gamma_e<\gamma_f$ and $m_e>m_f$, which indicates in conjunction with Eqs. (19)-(21) that the quasi-one-dimensional flow may be relatively a better approximation to the axially symmetric nozzle flow in the equilibrium situation than in the frozen situation. Sonic lines for an ideal dissociating diatomic (O_2) gas are shown in Fig. 1, where p_0, T_0, and D are, respectively, the stagnation pressure, the stagnation temperature and the dissociation energy of the gas.

When some nonuniformities of flow properties are introduced into the flowfield, the quantities ρ_*, p_*, a_*, and q_*, which have been introduced in nondimensionalizing flow variables, must be generally considered to be functions of β. The nonuniformities, however, appear in the system of basic equations only in terms of γ_i and p_*. This fact indicates that if γ_i and p_* are constant across the flowfield, the solution to the system for the nonuniform flow has just the same form as in the case of the uniform flow. In other words even if ρ_* and a_* are not constants but functions of β, Eqs. (11) to (18) constitute a solution to the system under the conditions of constant γ_i and p_*. From these discussions, it can be concluded that Hall's type of perturbation solutions can exist not only for uniform flows but also for nonuniform flows with constant γ_i and $p*$. Furthermore, closer investigation makes it clear that Hall's type of solution can easily be modified and extended to describe nonuniform transonic flows with variable γ_i under the condition of

$$p_*=\text{const} \quad (22)$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\beta}(\ln p_*)=O(\epsilon) \quad (23)$$

Boraas[2,3] has already considered the case of Eq. (22) for frozen flow and obtained an analytic solution to the system by applying Hall's technique. It is obvious that his result is not general in this respect because the condition of Eq. (22) is only one of the necessary conditions for the existence of Hall's type of perturbation solution.

Finally, it must be emphasized that the technique developed here can be applied in almost the same manner to flows of gases with multiple rate processes.

References

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[3] Boraas, S., "Transonic Nozzle Flow with Nonuniform Gas Properties," *AIAA Journal*, Vol. 11, Feb. 1973, pp. 210-215.